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Riemann Sums and Improper Integrals of Step Functions Related to the Prime Number Theorem

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1. The following result (as well as variations of it) is due to A. Wintner [8, pp. 685–686]:

THEOREM a. *Let ϕ be a real function, Riemann integrable on every $[\varepsilon, 1]$, $0 < \varepsilon < 1$. Suppose*[1] $\varepsilon \sum_{k=1}^{[1/\varepsilon]} \phi(k\varepsilon)$ *converges as* $\varepsilon \to 0+$. *Then the improper integral* $\int_{0+}^{1} \phi$ *converges and to the same limit.*

This result is contained implicitly in Theorem 3 of A. E. Ingham's paper [3]; cf. Section 1 of [4].

Theorem a, which looks quite innocent, is actually strongly connected with the Prime Number Theorem (P.N.T.). For its proof uses a fact leading in an elementary and simple way to the establishment of the P.N.T. Conversely, set, as usual, for every real $x \geqslant 1$,

$$\psi(x) = \sum \log p \tag{1}$$

where the sum is taken over all ordered pairs (p, m) for which p is a prime

[1] For a real x, $[x]$ is the integral part of x.

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and m a natural number satisfying $p^m \leqslant x$ (an "empty" sum is 0). It is well known that the P.N.T. follows in an elementary way from the relation

$$\lim_{x \to \infty} \psi(x)/x = 1. \tag{2}$$

As indicated in [4, Section 1], setting $\phi(x) \equiv \psi(x^{-1}) - x^{-1}$, one shows by elementary means that $\varepsilon \sum_{k=1}^{[1/\varepsilon]} \phi(k\varepsilon)$ converges as $\varepsilon \to 0+$. By Theorem a, $\int_{0+}^{1} \phi$ converges. But this implies, in an elementary way, the relation (2). Cf. also [8, p. 685].

2. Our purpose is to present a theorem similar to Theorem a but simpler, from which the P.N.T. readily follows. Instead of requiring Riemann integrability and studying sums based on partitions into subintervals of length ε, where ε varies continuously, we shall restrict ourselves to functions which are constant on each $(1/(n+1), 1/n]$, $n = 1, 2, \ldots$, and to Riemann sums based on partitions $(0, 1/n, 2/n, \ldots, 1)$, $n = 1, 2, \ldots$.

This theorem, like Theorem a, is of independent interest from the point of view of Real Analysis and Integration Theory and in Sections 3–11 we shall study it and related results from that point of view without recourse to Theorem a. It is

THEOREM I. *Let f be a real step function*:

$$\left.\begin{aligned} &f(x) = a_n \textit{ throughout } (1/(n+1), 1/n], \qquad n = 1, 2, \ldots, \\ &\textit{namely}, \\ &f(x) = a_{[1/x]} \textit{ throughout } (0, 1]. \end{aligned}\right\} \tag{3}$$

Suppose the special sequence of Riemann sums

$$B_n = (1/n) \sum_{k=1}^{n} f(k/n), \qquad n = 1, 2, \ldots, \tag{4}$$

converges. Then so does the improper Riemann integral $\int_{0+}^{1} f$, and to the same limit.

To derive from Theorem I the P.N.T., set, with (1),

$$f(x) \equiv \psi(x^{-1}) - [x^{-1}]. \tag{5}$$

Given mappings g, h of the natural numbers into the reals, we denote, as usual,

$$(g * h)(k) = \sum_{j \mid k, j \geqslant 1} g(j) h(k/j), \qquad k = 1, 2, \ldots, \tag{6}$$

so that [2, p. 559, (2.5)]

$$\sum_{k=1}^{n} (g * h)(k) = \sum_{k=1}^{n} \sum_{j=1}^{[n/k]} g(j)\, h(k), \qquad n = 1, 2,\ldots. \tag{7}$$

Denoting by 1 the constant function 1, we have by (6), for $k = 1, 2,\ldots$, $(1 * 1)(k) = d(k)$, the number of positive divisors of k. Hence, by (7),

$$\sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} (1 * 1)(k) = \sum_{k=1}^{n} \left[\frac{n}{k}\right], \qquad n = 1, 2,\ldots.$$

A classical result of Dirichlet [2, p. 560, (2.7)] therefore yields, for $n = 1, 2,\ldots$ (γ being Euler's constant),

$$\sum_{k=1}^{n} \left[\frac{n}{k}\right] = n \log n + (2\gamma - 1)n + O(\sqrt{n}). \tag{8}$$

We shall use also the formula [2, p. 559, (2.6)]

$$\sum_{k=1}^{n} \psi(n/k) = n \log n - n + O(1 + \log n), \qquad n = 1, 2,\ldots. \tag{9}$$

Now (4) applied to (5) gives, in view of (8) and (9), $B_n \to -2\gamma$. Hence Theorem I implies that $\int_{0+}^{1} f$ converges. According to the end of Section 1, to obtain an elementary proof of the P.N.T. it is enough to provide an elementary proof that $\int_{0+}^{1} \phi$ converges, where $\phi(x) \equiv \psi(x^{-1}) - x^{-1}$. This convergence, in turn, follows at once by the fact that

$$\begin{aligned}\int_{0+}^{1} (f - \phi) &= \int_{0+}^{1} (x^{-1} - [x^{-1}])\, dx = \lim_{n\to\infty} \int_{1/n}^{1} (x^{-1} - [x^{-1}])\, dx \\ &\lim_{n\to\infty} \sum_{k=2}^{n} \int_{1/k}^{1/(k-1)} (x^{-1} - [x^{-1}])\, dx \\ &= \lim_{n\to\infty} \sum_{k=2}^{n} \log k - \log(k-1) - (1/k) = 1 - \gamma.\end{aligned}$$

Thus, an elementary proof of Theorem I (even only for some class of functions including (5)) will yield a new elementary proof of the P.N.T.

A derivation of Theorem I from Theorem a is given in Section 12.

3. We shall assume henceforth (3) with real a_n and investigate the relationship between convergence of $\int_{0+}^{1} f$ and that of B_n. In this section we make some simple observations.

LEMMA 1. *The improper integral* $\int_{0+}^{1} f$ *converges iff the sequence* $\int_{1/n}^{1} f$ *does.*

Observe that such a result does not hold in general, even for a step function. Consider, e.g., the function F defined on $(0, 1]$ as follows. Let $x \in (1/(n+1), 1/n]$, n a positive integer, and let x_n be the midpoint of that interval. If $x \in (1/(n+1), x_n]$, we set $F(x) = n^2$; otherwise, $F(x) = -n^2$. Then $\int_{1/n}^{1} F = 0$ for $n = 1, 2, \ldots$, but clearly $\int_{0+}^{1} F$ diverges.

Proof of Lemma 1. Suppose $\int_{1/n}^{1} f$ converges to L. Let $\varepsilon > 0$. Let n_0 be an integer $\geqslant 1$ such that

$$\left| L - \int_{1/n}^{1} f \right| < \varepsilon \qquad \text{whenever} \quad n \geqslant n_0.$$

Suppose $0 < \delta < 1/n_0$. We shall show that $|L - \int_{\delta}^{1} f| < \varepsilon$. Let $\delta \in [1/(n_1 + 1), 1/n_1)$, n_1 a positive integer $\geqslant n_0$. Then $\int_{\delta}^{1} f$ lies between $\int_{1/n_1}^{1} f$ and $\int_{1/(n_1+1)}^{1} f$. Since the last two integrals differ from L by less than ε, so does $\int_{\delta}^{1} f$.

Since, for $n = 1, 2, \ldots$, $\int_{1/(n+1)}^{1} f = \sum_{k=1}^{n} \int_{1/(k+1)}^{1/k} f = \sum_{k=1}^{n} a_k/[k(k+1)]$, the convergence of $\int_{0+}^{1} f$ is equivalent to that of $\sum_{k=1}^{\infty} a_k/[k(k+1)]$.

THEOREM 1. *Suppose* $(a_n)_{n=1}^{\infty}$ *is monotone. Then* B_n *converges iff* $\int_{0+}^{1} f$ *converges, in which case* $\lim_{n \to \infty} B_n = \int_{0+}^{1} f$.

Proof. The claims follow from the theorem that if a real function F is monotone on $(0, 1]$, then $(1/n) \sum_{k=1}^{n} F(k/n)$ converges iff $\int_{0+}^{1} F$ does, in which case both limits are equal (compare [1, pp. 222–225] and [7, p. 79]).

4. THEOREM 2. *Suppose, throughout* $(0, 1]$, $|f| \leqslant g$ *where* g *is a real function, monotone nonincreasing on* $(0, 1]$, *with* $\int_{0+}^{1} g < \infty$. *Then* $B_n \to \int_{0+}^{1} f$.

Proof. By Theorem 3 and Definition 4 of [5], f is dominantly integrable. (In that paper "decreasing" means "nonincreasing".) Hence, by Theorem 3 of [6], for every Q-sequence $(\Phi_n)_{n=1}^{\infty}$ corresponding to $g(t) \equiv t$, $\Phi_n(f) \to \int_{0+}^{1} f$. Perhaps the simplest such Φ_n is the arithmetic mean of the values of the function at $1/n, 2/n, \ldots, n/n$ (take, in Definition 1 of [6], $g(x) \equiv 1$, $\delta = 1/2$, $d(n) \equiv n$, $c_j^{(n)} \equiv 1$, $t_j^{(n)} \equiv j/n$, $\tau_j^{(n)} \equiv j/n$, $B = 1$ and $M = 2$). Thus $B_n \to \int_{0+}^{1} f$.

EXAMPLE 1. Let $0 \leqslant \alpha < 1$ and suppose $a_n = O(n^{\alpha})$. Then, for some constant c and all $x \in (0, 1]$, $|f(x)| = |a_{[1/x]}| \leqslant c[1/x]^{\alpha} \leqslant cx^{-\alpha}$. By Theorem 2, $B_n \to \int_{0+}^{1} f$.

5. From (4) and (3),

$$B_n = (1/n) \sum_{k=1}^{n} a_{[n/k]}, \qquad n = 1, 2, \ldots. \tag{10}$$

Given integers $1 \leqslant j \leqslant n$, let $\alpha_j(n)$ be the number of integers k for which $[n/k] = j$. By (10),

$$B_n = (1/n) \sum_{j=1}^{n} \alpha_j(n) a_j, \qquad n = 1, 2, \ldots. \tag{11}$$

Observe that for integers $1 \leqslant j \leqslant n$, $\alpha_j(n)$ is the number of integers in $(n/(j+1), n/j]$, namely,

$$\alpha_j(n) = [n/j] - [n/(j+1)]. \tag{12}$$

Set

$$A_n = \sum_{j=1}^{n} a_j/[j(j+1)], \qquad n = 1, 2, \ldots, \tag{13}$$

so that, by the sentence preceding Theorem 1, $\int_{0+}^{1} f$ converges iff A_n does, in which case $A_n \to \int_{0+}^{1} f$.

(13) and (12) readily yield, for integers $1 \leqslant n_1 \leqslant n$,

$$\left| A_{n_1} - (1/n) \sum_{j=1}^{n_1} \alpha_j(n)\, a_j \right| \leqslant (1/n) \sum_{j=1}^{n_1} |a_j|. \tag{14}$$

THEOREM 3. *Suppose a_n is bounded below or above and B_n converges. Then so does A_n. (See also Theorem 4.)*

Proof. We may assume a_n is bounded below (otherwise, consider $-a_n$) and, in fact, by 0 (if by some a, consider $a_n - a$). Suppose $A_n \to \infty$. Choose $n_1 \geqslant 1$ with $A_{n_1} \geqslant B + 2$, where $B = \lim_{n\to\infty} B_n$. Let $n_2 > n_1$ be an integer $> \sum_{j=1}^{n_1} |a_j|$. If $n \geqslant n_2$, then by (11) and (14),

$$\begin{aligned} B + 2 - B_n &\leqslant A_{n_1} - B_n \\ &= A_{n_1} - (1/n) \sum_{j=1}^{n_1} \alpha_j(n) a_j - (1/n) \sum_{j=n_1+1}^{n} \alpha_j(n) a_j < 1, \end{aligned}$$

so that $B_n > B + 1$, a contradiction.

6. Theorem 3 can be strengthened.

THEOREM 4. *Assume the hypotheses of Theorem 3. Then $A_n \to \lim_{n\to\infty} B_n$.*

To prove Theorem 4 we need

LEMMA 2. *For $n = 1, 2, \ldots$, set*

$$\lambda_j(n) = (\{n/(j+1)\} - \{n/j\})/n, \qquad j = 1, 2, \ldots, n, \tag{15}$$

where, for every real x, $\{x\}$ is its fractional part $x - [x]$. Suppose each $a_n \geqslant 0$, A_n converges and so does

$$L_n \equiv \sum_{j=1}^{n} \lambda_j(n) a_j. \tag{16}$$

Then $\lim_{n\to\infty} L_n = 0$.

Proof of Lemma 2. Suppose not. Then for some $b > 0$, $|L_n| \geqslant b$ for all $n \geqslant$ some $n_0 \geqslant 1$. Hence $\sum_{n=2}^{\infty} |L_n/(n \log n)|$ diverges, as $\sum_{n=2}^{\infty} 1/(n \log n)$ does. We shall therefore prove that $\sum_{n=2}^{\infty} |L_n/(n \log n)|$ converges.

We first show

$$\sum_{n=j}^{\infty} |\lambda_j(n)|/(n \log n) = O(j^{-2}). \tag{17}$$

Since, for $j \geqslant 2$, $\lambda_j(j) = 1/(j+1)$ and

$$\sum_{n=j^2}^{\infty} |\lambda_j(n)|/(n \log n) < \sum_{n=j^2}^{\infty} n^{-2} < \int_{j^2-1}^{\infty} dx/x^2 < 2/j^2,$$

it is enough to prove that

$$\sum_{n=j+1}^{j^2-1} |\lambda_j(n)|/(n \log n) = O(j^{-2}). \tag{18}$$

For $j \geqslant 3$ set

$$\sum_{n=j+1}^{j^2-1} |\lambda_j(n)|/(n \log n) = \sum_j{}' + \sum_j{}''$$

where

$$\sum_j{}' = \sum_{k=1}^{j-1} \sum_{n=k(j+1)}^{(k+1)j-1} |\lambda_j(n)|/(n \log n),$$

$$\sum_j{}'' = \sum_{k=1}^{j-2} \sum_{n=(k+1)j}^{(k+1)(j+1)-1} |\lambda_j(n)|/(n \log n).$$

If $j \geqslant 2$, $1 \leqslant k \leqslant j-1$ and $k(j+1) \leqslant n \leqslant (k+1)j - 1$, we can set

$$n = kj + m = k(j+1) + m - k, \qquad 0 < m < j, 0 \leqslant m - k < j + 1,$$

so that, by (15), $\lambda_n(j) = -1/[j(j+1)]$. Hence, if $j \geqslant 2$,

$$\left| \sum_j{}' \right| = [j(j+1)]^{-1} \sum_{k=1}^{j-1} \sum_{n=k(j+1)}^{(k+1)j-1} (n \log n)^{-1}$$
$$\leqslant [j(j+1)]^{-1} \sum_{n=j+1}^{j^2-1} (n \log n)^{-1}$$
$$< [j(j+1)]^{-1} \int_j^{j^2-1} (x \log x)^{-1} dx$$
$$< j^{-2} \log \log x \Big|_j^{j^2} = j^{-2} \log 2.$$

If $j \geqslant 3$, then

$$\left| \sum_j{}'' \right| < \sum_{k=1}^{j-2} \sum_{n=(k+1)j}^{(k+1)(j+1)-1} (n^2 \log n)^{-1}$$
$$< \sum_{k=1}^{j-2} (k+1)[(k+1)j]^{-2} \log^{-1}[(k+1)j]$$
$$< (j^2 \log j)^{-1} \sum_{k=2}^{j-1} k^{-1} < (j^2 \log j)^{-1} \int_1^j dx/x = j^{-2}.$$

So (18) and hence (17) are established.
Now, for every $N \geqslant 2$,

$$\sum_{n=2}^{N} |L_n/(n \log n)| \leqslant \sum_{n=2}^{N} (n \log n)^{-1} \sum_{j=1}^{n} |\lambda_j(n)|\, a_j$$
$$= a_1 \sum_{n=2}^{N} |\lambda_1(n)|\, (n \log n)^{-1}$$
$$+ \sum_{j=2}^{N} a_j \sum_{n=j}^{N} |\lambda_j(n)|\, (n \log n)^{-1}$$
$$\leqslant a_1 \sum_{n=2}^{\infty} (n^2 \log n)^{-1} + \alpha \sum_{j=1}^{\infty} a_j/[j(j+1)],$$

α being some constant, which completes the proof of the Lemma.

Proof of Theorem 4. As in the proof of Theorem 3, we may assume each $a_n \geqslant 0$. For $n = 1, 2,\ldots$, by (11), (12), (13), (15), and (16),

$$\left.\begin{aligned} B_n - A_n &= (1/n) \sum_{j=1}^{n} ([nj^{-1}] - [n(j+1)^{-1}] - (nj^{-1} - n(j+1)^{-1}))a_j \\ &= \sum_{j=1}^{n} \lambda_j(n) a_j = L_n. \end{aligned}\right\} \quad (19)$$

By Theorem 3 and Lemma 2, $L_n \to 0$. Hence $A_n \to \lim_{n\to\infty} B_n$.

7. THEOREM 5. *For every $\delta \in (0, 1)$ let $V(\delta)$ denote the total variation of f on $[\delta, 1]$ (which is clearly finite). Suppose $\lim_{n\to\infty}(1/n)\, V(1/n) = 0$. Then A_n converges iff B_n does, in which case*

$$\lim_{n\to\infty} A_n = \int_{0+}^{1} f = \lim_{n\to\infty} B_n.$$

Proof. By (19), $B_n - A_n \equiv L_n$. So it is enough to show $L_n \to 0$. But by (16) and (15), for $n = 2, 3,\ldots,$

$$\begin{aligned} |L_n| &= (1/n) \left| (n/(n+1))a_n + \sum_{j=2}^{n} \{n/j\}(a_{j-1} - a_j) \right| \\ &\leqslant |a_n/(n+1)| + (1/n) \sum_{j=2}^{n} |a_j - a_{j-1}| \\ &\leqslant (1/n) \left(|a_1| + |a_n - a_1| + \sum_{j=2}^{n} |a_j - a_{j-1}| \right) \\ &\leqslant (1/n)(|a_1| + 2V(1/n)) \to 0. \end{aligned}$$

EXAMPLE 2. Let $a_n \equiv n/\log(n+1)$, so that A_n diverges. Then, for $n = 2, 3,\ldots, a_n > a_{n-1}$ so that $V(1/n) = (n/\log(n+1)) - (1/\log 2)$ and, hence, $(1/n)\, V(1/n) \to 0$. Therefore, by Theorem 5, B_n diverges.

EXAMPLE 3. Let $a_n = n$ when $n = 2^k$, $k = 0, 1, 2,\ldots, a_n = 0$ otherwise. Then

$$A_n \to \sum_{j=1}^{\infty} a_j/[j(j+1)] = \sum_{k=0}^{\infty} 2^k/[2^k(2^k+1)] = \sum_{k=0}^{\infty} 1/(2^k+1)$$

but B_n diverges (see Section 9 below). On the other hand, $\delta V(\delta)$ is bounded in $(0, 1)$. For let $\delta \in (0, 1)$, say $2^{-k-1} < \delta \leqslant 2^{-k}$, k an integer $\geqslant 0$. Then $V(\delta) < 2 \sum_{j=0}^{k} 2^j = 2(2^{k+1} - 1)$ and, hence, $\delta V(\delta) < 4$. Thus the relation $(1/n)\, V(1/n) \to 0$ in Theorem 5 cannot be replaced by the boundedness of $\delta V(\delta)$ in $(0, 1)$.

8. DEFINITION 1. Condition C is the following property: For every $\varepsilon > 0$ there is an integer $n_0(\varepsilon) \geqslant 1$ such that for each integer $n_1 \geqslant n_0(\varepsilon)$ there is an integer $m_0(\varepsilon, n_1) > n_1$ so that, if $n \geqslant m_0(\varepsilon, n_1)$, then

$$\left| (1/n) \sum_{j=n_1+1}^{n} \alpha_j(n) a_j \right| < \varepsilon.$$

THEOREM 6. *Assume Condition C. Then A_n converges iff B_n does, in which case* $\lim_{n\to\infty} A_n = \int_{0+}^{1} f = \lim_{n\to\infty} B_n$.

Proof. Suppose A_n converges, say to A. Let $\varepsilon > 0$. Choose $n_\varepsilon \geqslant 1$ such that $|A - A_n| < \varepsilon/3$ if $n \geqslant n_\varepsilon$. Using Definition 1, set

$$n_1 = \max(n_0(\varepsilon/3), n_\varepsilon), \qquad m = m_0(\varepsilon/3, n_1).$$

Let m^* be an integer $\geqslant m$ such that if $n \geqslant m^*$, then the right hand side of (14) is $<\varepsilon/3$. If $n \geqslant m^*$, then

$$|A - B_n| \leqslant |A - A_{n_1}| + \left| A_{n_1} - (1/n) \sum_{j=1}^{n_1} \alpha_j(n) a_j \right| + \left| (1/n) \sum_{j=n_1+1}^{n} \alpha_j(n) a_j \right| < (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon.$$

Suppose B_n converges, say to B. Let $\varepsilon > 0$. Choose $\nu_\varepsilon \geqslant 1$ such that $|B - B_n| < \varepsilon/3$ if $n \geqslant \nu_\varepsilon$. Referring to Definition 1, let n_1 be an integer $\geqslant n_0(\varepsilon/3)$ and set $\mu = m_0(\varepsilon/3, n_1)$. Let μ^* be a positive integer such that if $n \geqslant \mu^*$, then the right hand side of (14) is $<\varepsilon/3$. Set, finally, $n^* = \max(\nu_\varepsilon, \mu, \mu^*)$. Then

$$|B - A_{n_1}| \leqslant |B - B_{n^*}| + \left| A_{n_1} - (1/n^*) \sum_{j=1}^{n_1} \alpha_j(n^*) a_j \right| + \left| (1/n^*) \sum_{j=n_1+1}^{n^*} \alpha_j(n^*) a_j \right| < (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon.$$

THEOREM 7. *Suppose A_n and B_n converge and to the same limit A. Then Condition C holds.*

Proof. Let $\varepsilon > 0$. Let $n_0(\varepsilon) \geqslant 1$ be an integer such that $|A_n - A| < \varepsilon/3$ and $|B_n - A| < \varepsilon/3$ whenever $n \geqslant n_0(\varepsilon)$. For every integer $n_1 \geqslant n_0(\varepsilon)$, let $m_0(\varepsilon, n_1)$ be an integer $>n_1$ such that if $n \geqslant m_0(\varepsilon, n_1)$, then the right hand side of (14) is $<\varepsilon/3$. If integers n_1, n satisfy $n_1 \geqslant n_0(\varepsilon)$, $n \geqslant m_0(\varepsilon, n_1)$, then

$$\left|(1/n)\sum_{j=n_1+1}^{n}\alpha_j(n)a_j\right|$$
$$\leqslant |B_n - A| + |A - A_{n_1}| + \left|A_{n_1} - (1/n)\sum_{j=1}^{n_1}\alpha_j(n)a_j\right|$$
$$< (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon.$$

THEOREM 8. *Let* $(1/n)\sum_{j=1}^{n}\alpha_j(n)|a_j|$ *converge. Then so do* A_n *and* B_n, *and* $\lim_{n\to\infty}A_n = \lim_{n\to\infty}B_n$.

Proof. By Theorems 4 and 7, Condition C, *applied to* $|a_n|$, holds. Hence so does Condition C itself. By Theorem 4, $\sum_{j=1}^{\infty}a_j/[j(j+1)]$ converges (absolutely). By Theorem 6, B_n converges to that infinite sum.

9. We return to Example 3 and prove that B_n diverges. Let $k \geqslant 4$ be an even integer. By (12),

$$\alpha_1(2^k) = 2^{k-1} = \alpha_1(2^k - 1).$$

Let j be an integer. If $k/2 \leqslant j \leqslant k$, then $\alpha_{2^j}(2^k) = 1$, while if $k/2 \leqslant j \leqslant k-1$, then $2^k - 1 \geqslant 2^k + 2^{k-j} - 2^j - 1 = (2^j + 1)(2^{k-j} - 1)$ and hence

$$2^{k-j} - 1 \leqslant (2^k - 1)/(2^j + 1) < (2^k - 1)/2^j < 2^{k-j};$$

so there are no integers in $((2^k - 1)/(2^j + 1),\ (2^k - 1)/2^j]$ and therefore $\alpha_{2^j}(2^k - 1) = 0$. Hence

$$B_{2^k} = 2^{-k}\sum_{i=1}^{2^k}\alpha_i(2^k)a_i = 2^{-k}\sum_{j=0}^{k}\alpha_{2^j}(2^k)2^j$$
$$> 2^{-k}\left(2^{k-1} + \sum_{j=1}^{(k/2)-1}(2^{k-j} - 1 - 2^k(2^j + 1)^{-1})2^j + \sum_{j=k/2}^{k}2^j\right),$$
$$B_{2^k-1} = (2^k - 1)^{-1}\sum_{i=1}^{2^k-1}\alpha_i(2^k - 1)a_i = (2^k - 1)^{-1}\sum_{j=0}^{k-1}\alpha_{2^j}(2^k - 1)2^j$$
$$< (2^k - 1)^{-1}\left(2^{k-1} + \sum_{j=1}^{(k/2)-1}((2^k - 1)2^{-j} - (2^k - 1)(2^j + 1)^{-1} + 1)2^j\right),$$

and

$$B_{2^k} - B_{2^k-1} > 2^{k-1}(2^{-k} - (2^k - 1)^{-1}) - (2^{-k} + (2^k - 1)^{-1})\sum_{j=1}^{(k/2)-1}2^j$$
$$+ 2^{-k}\sum_{j=k/2}^{k}2^j$$
$$= (5/2) - (2^{k-1}(2^k - 1)^{-1} + (2^{-k} + (2^k - 1)^{-1})(2^{k/2} - 2) + 2^{-k/2})$$

which $\to 2$ as $k \to \infty$. Hence, B_n is not a Cauchy sequence and therefore diverges.

10. Consider an arbitrary real sequence $(a_n)_{n=1}^{\infty}$ and a prime $p \geqslant 3$. Since $[(p-1)/k] = [p/k]$ for $k = 2, 3, \ldots, p-1$, we have by (10),

$$\left.\begin{aligned} B_p - B_{p-1} &= p^{-1}\left(a_p + \sum_{k=2}^{p-1} a_{[p/k]} + a_1\right) - (p-1)^{-1}\left(a_{p-1} + \sum_{k=2}^{p-1} a_{[p/k]}\right) \\ &= p^{-1}(a_1 + a_p) - (p-1)^{-1}a_{p-1} + (p^{-1} - (p-1)^{-1}) \sum_{k=2}^{p-1} a_{[p/k]}. \end{aligned}\right\} \tag{20}$$

Using this observation, we give a simple example of an alternating a_n for which A_n converges but B_n does not.

EXAMPLE 4. Let $a_n = (-1)^n n$, $n = 1, 2, \ldots$, so that $A_n \to -1 + \log 2$. For $n = 3, 4, \ldots$, set $b_n = \sum_{k=2}^{n-1} a_{[n/k]}$ so that

$$|b_n| \leqslant \sum_{k=2}^{n-1} \left[\frac{n}{k}\right] \leqslant n \sum_{k=2}^{n-1} k^{-1} < n \log(n-1). \tag{21}$$

By (20) and (21), for every prime $p \geqslant 3$,

$$B_p - B_{p-1} = -p^{-1} - 1 - 1 - [p(p-1)]^{-1} b_p,$$
$$|B_p - B_{p-1}| > 2 - (p(p-1))^{-1} |b_p| > 2 - (p-1)^{-1} \log(p-1).$$

Thus B_n is not a Cauchy sequence and hence diverges.

11. THEOREM 9. *Suppose $(a_n)_{n=1}^{\infty}$ is monotone and either B_n or $\int_{0+}^{1} f$ converges (see Theorem 1). Then $a_n/n \to 0$.*

Proof. We may assume $(a_n)_{n=1}^{\infty}$ is nondecreasing (otherwise, consider $(-a_n)_{n=1}^{\infty}$). We may also assume each a_n is $\geqslant 0$ (otherwise, consider $(a_n - a_1)_{n=1}^{\infty}$). Then, for $n = 1, 2, \ldots$,

$$a_n/n \leqslant 2(n+1)\, a_n [n(2n+1)]^{-1} \leqslant 4 \sum_{k=n}^{2n} a_k [k(k+1)]^{-1} \to 0.$$

THEOREM 10. *Suppose a_n is bounded below or above and B_n converges. Then $a_n/n \to 0$.*

Proof. As in the proof of Theorem 3 we may assume $a_n \geqslant 0$, $n = 1, 2, \ldots$.

By Theorems 4 and 7, Condition C holds. Let $\varepsilon > 0$. By Definition 1, if $n \geqslant m_0\ (\varepsilon, n_0(\varepsilon))$, then

$$a_n/n \leqslant (1/n) \sum_{j=n_0(\varepsilon)+1}^{n} \alpha_j(n) a_j < \varepsilon.$$

THEOREM 11. *Suppose* $0 \leqslant b_n \leqslant b_{n+1}$, $a_n \geqslant -b_n$ *for* $n = 1, 2, \ldots$, $\beta = \sum_{k=1}^{\infty} b_k/[k(k+1)] < \infty$ *and* B_n *converges. Then* $B_n \to \int_{0+}^{1} f$ *and* $a_n/n \to 0$.

Proof. For $n = 1, 2, \ldots$; let $c_n = a_n + b_n \geqslant 0$. By Theorem 1, $n^{-1} \sum_{k=1}^{n} b_{[n/k]} \to \beta$. By Theorem 4, $\sum_{k=1}^{\infty} c_k/[k(k+1)] = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} c_{[n/k]} = \lim_{n\to\infty} B_n + \beta$. Hence $\int_{0+}^{1} f = \sum_{k=1}^{\infty} a_k/[k(k+1)] = \lim_{n\to\infty} B_n$. By Theorem 10, $b_n/n \to 0$, $c_n/n \to 0$. Hence $a_n/n \to 0$.

12. We derive now Theorem I of Section 2 from Theorem a of Section 1. Note that Theorem a has not yet been proved in an elementary way.

By Theorem a, it is enough to prove that $\lim_{\varepsilon\to 0+} \varepsilon \sum_{k=1}^{[1/\varepsilon]} f(k\varepsilon) = B$, where $B = \lim_{n\to\infty} B_n$. Let $h_1, h_2, \ldots \in (0, 1]$ and satisfy $h_n \to 0$. We shall prove that $\lim_{n\to\infty} h_n \sum_{k=1}^{[h_n^{-1}]} f(kh_n) = B$. For $n = 1, 2, \ldots$,

$$\begin{aligned}
&\left| B_{[h_n^{-1}]} - h_n \sum_{k=1}^{[h_n^{-1}]} f(kh_n) \right| \\
&= \left| [h_n^{-1}]^{-1} \sum_{k=1}^{[h_n^{-1}]} a_{[k^{-1}[h_n^{-1}]]} - h_n \sum_{k=1}^{[h_n^{-1}]} a_{[k^{-1}h_n^{-1}]} \right| \\
&= ([h_n^{-1}]^{-1} - h_n) \cdot \left| \sum_{k=1}^{[h_n^{-1}]} a_{[k^{-1}[h_n^{-1}]]} \right| \\
&\leqslant h_n [h_n^{-1}]^{-1} \cdot \left| \sum_{k=1}^{[h_n^{-1}]} a_{[k^{-1}[h_n^{-1}]]} \right| = h_n |B_{[h_n^{-1}]}| \to 0.
\end{aligned}$$

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